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Hubbert, Simon; Morton, Tanya M. (2004) L_p-error estimates for radial basis function interpolation on the sphere. *Journal of Approximation Theory* **129** (1): 58-77

L_p−error estimates for radial basis function interpolation on the sphere

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Abstract

In this paper we review the variational approach to radial basis function interpolation on the sphere and establish new L_p−error bounds, for $p \in [1, \infty]$. These bounds are given in terms of a measure of the density of the interpolation points, the dimension of the sphere and the smoothness of the underlying basis function.

1   Introduction

We shall study the radial basis function method [3] for solving the following spherical interpolation problem.

Problem 1.1.   *Given a set, $\Xi = \{\xi_1, \ldots, \xi_N\}$, of N distinct data points on S^{d-1}, and a target function $f : S^{d-1} \to \mathbb{R}$, find a function $s : S^{d-1} \to \mathbb{R}$ that satisfies the conditions*

$$s(\xi_i) = f(\xi_i), \quad i = 1, \ldots, N. \tag{1.1}$$

The accuracy of the method will certainly depend upon the distribution of the data points $\Xi \subset S^{d-1}$, and so we assign a density measure (mesh-norm) by

$$h = \sup_{\eta \in S^{d-1}} \min\{g(\eta, \xi_i) : \xi_i \in \Xi\}, \tag{1.2}$$

where g denotes the geodesic metric given by $g(\xi, \eta) = \cos^{-1}(\xi^T \eta)$, for $\xi, \eta \in S^{d-1}$. One would expect that, as the points become more and more dense, i.e, as $h \to 0$, then the interpolation error should decrease. This intuition is justified since, for sufficiently smooth target functions f, we will prove results of the form

$$\|s - f\|_{L_p(S^{d-1})} = O(h^{\lambda_p}), \text{ where } \lambda_p > 0, \text{ and } p \in [1, \infty].$$

Before we can analyse the method further it is necessary to review some basic Fourier theory for the sphere. We begin in $\mathbb{R}^d$ where we say that a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ of degree $k \geq 0$, is

(i) **harmonic** if it satisfies the Laplace equation

$$\Delta_x p(x) = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \right) p(x) = 0, \tag{1.3}$$

(ii) **k-homogeneous** if $p(tx) = t^k p(x)$ for any $t > 0$.

We let $\mathcal{HP}_k(\mathbb{R}^d)$ denote the space of all polynomials of degree k on $\mathbb{R}^d$ that are both harmonic and $k-$homogeneous. The restriction of this class to the unit sphere is of particular interest.

Definition 1.2. *Let $p_k \in \mathcal{HP}_k(\mathbb{R}^d)$, then its restriction to the sphere, $\mathcal{Y}_k = p_k \mid_{S^{d-1}}$, is a spherical harmonic of order k on S^{d-1}. We let $\mathcal{H}_k^*(S^{d-1})$ denote the space of spherical harmonics of exact order k.*

The following result follows from an application of Greens theorem, see [16].

Theorem 1.3. *The exact order spherical harmonic spaces $\mathcal{H}_k^*(S^{d-1})$ and $\mathcal{H}_l^*(S^{d-1})$ with $k \neq l$ are $L_2(S^{d-1})-$orthogonal.*

In view of this we let $\mathcal{H}_k(S^{d-1}) = \oplus_{j=0}^k \mathcal{H}_j^*(S^{d-1})$ denote the space of spherical harmonics of order at most k.

For every $x \in \mathbb{R}^d \setminus \{0\}$ we can write $x = r\xi$, where $r = \|x\|$ and $\xi \in S^{d-1}$. This observation allows us to rewrite the Laplace operator as follows

$$\Delta_x = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\xi^*, \quad \text{for } x \in \mathbb{R}^d \setminus \{0\} \tag{1.4}$$

where Δ_ξ^*, having no radial component, is the Laplace operator for the sphere, see [16]. Let $p_k \in \mathcal{HP}_k(\mathbb{R}^d)$ then, for $x \in \mathbb{R}^d \setminus \{0\}$, we can write $p_k(x) = r^k \mathcal{Y}_k(\xi)$. Applying (1.4) gives

$$0 = \Delta_x p_k(x) = k(k+d-2)r^{k-2}\mathcal{Y}_k(\xi) + r^{k-2}\Delta_\xi^* \mathcal{Y}_k(\xi)$$

$$\implies \Delta_\xi^* \mathcal{Y}_k(\xi) + \lambda_k \mathcal{Y}_k(\xi) = 0, \quad \text{where} \quad \lambda_k = k(k+d-2). \tag{1.5}$$

The following result is taken from [6] Chapter 3.

Theorem 1.4. *The space $\mathcal{H}_k^*(S^{d-1})$ is precisely the eigenspace of Δ_ξ^* corresponding to the eigenvalue λ_k. Furthermore, the direct sum of these eigenspaces is all of $L_2(S^{d-1})$, that is*

$$L_2(S^{d-1}) = \oplus_{k=0}^\infty \mathcal{H}_k^*(S^{d-1}). \tag{1.6}$$

The following theorem illustrates the role of the $\mathcal{H}_k^*(S^{d-1})$ spaces in the context of spherical Fourier analysis.

Theorem 1.5. *Assume $d \geq 2$ and let $\mathcal{B}_k = \{\mathcal{Y}_{k,l} : l = 1, \ldots, N_{k,d}\}$ denote an orthonormal basis for $\mathcal{H}_k^*(S^{d-1})$. Then, in view of (1.6), the orthonormal system $\{\mathcal{B}_k\}_{k \geq 0}$ is complete in $L_2(S^{d-1})$, and every $f \in L_2(S^{d-1})$ has a spherical Fourier expansion given by*

$$f = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{f}_{k,l} \mathcal{Y}_{k,l} \tag{1.7}$$

where

$$\hat{f}_{k,l} = (f, \mathcal{Y}_{k,l})_{L_2(S^{d-1})} = \int_{S^{d-1}} f(\xi) \mathcal{Y}_{k,l}(\xi) d\omega_{d-1}(\xi), \tag{1.8}$$

are the spherical Fourier coefficients of f.

The following result is the famous addition theorem for spherical harmonics [16].

Theorem 1.6. *Let ω_{d-1} denote the surface area of S^{d-1}, and let $\{\mathcal{Y}_{k,l} : l = 1, \ldots, N_{k,d}\}$ be an orthonormal basis for $\mathcal{H}_k^*(S^{d-1})$. Then, for any $\xi, \eta \in S^{d-1}$, the function*

$$P_{k,d}(\xi^T \eta) := \frac{\omega_{d-1}}{N_{k,d}} \sum_{l=1}^{N_{k,d}} \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta), \tag{1.9}$$

is a unique real valued, univariate polynomial of degree k defined on $[-1, 1]$.

The polynomials $P_{k,d}$, $k \geq 0$, are commonly called the $d-$*dimensional Legendre polynomials* and they play an important role in this paper. In view of this we collect together some of their key properties.

$\diamond$ **Orthogonality:** $P_{k,d}$ is a polynomial of degree k, such that $P_{k,d}(1) = 1$, $|P_{k,d}(t)| \leq 1$ and the orthogonality relation is

$$\int\limits_{-1}^{1} P_{j,d}(t) P_{k,d}(t) (1 - t^2)^{\frac{d-3}{2}} dt = \frac{\omega_{d-1}}{\omega_{d-2} N_{k,d}} \delta_{jk}. \tag{1.10}$$

$\diamond$ **Legendre-Fourier Expansion:** Let ψ be continuous on the interval $[-1, 1]$ such that

$$\int_{-1}^{1} |\psi(t)| (1 - t^2)^{\frac{d-3}{2}} dt < \infty, \tag{1.11}$$

then ψ has a Legendre series expansion $\psi(t) = \sum_{k=0}^{\infty} a_k P_{k,d}(t)$. Furthermore, ψ induces a zonal kernel $\Psi(\xi, \eta) = \psi(\xi^T \eta)$ which has a spherical Fourier series of the form,

$$\Psi(\xi, \eta) = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta), \tag{1.12}$$

where $\{\hat{c}_k\}_{k=0}^{\infty}$ denote the spherical Fourier coefficients of Ψ, and, using (1.9) and (1.10), these are given by

$$\hat{c}_k = \frac{a_k \omega_{d-1}}{N_{k,d}} = \omega_{d-2} \int_{-1}^{1} P_{k,d}(t) \psi(t) (1-t^2)^{\frac{d-3}{2}} dt. \tag{1.13}$$

$\diamond$ **Sobolev spaces:** Let λ_k be as in (1.5). The Sobolev space $W_2^{\beta}(S^{d-1})$ of order $\beta \geq 0$, is defined to be the Hilbert space of functions $f \in L_2(S^{d-1})$ with norm

$$\|f\|^2_{W_2^{\beta}(S^{d-1})} = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} (1+\lambda_k)^{\beta} |\hat{f}_{k,l}|^2.$$

2   The radial basis function method for the sphere

One way of solving Problem 1.1 is to choose the interpolant s from the linear space spanned by the N functions

$$\xi \mapsto \psi(g(\xi, \xi_j)), \quad j = 1, \ldots, N,$$

where g denotes the geodesic metric on S^{d-1}, and $\psi : [0, \pi] \to \mathbb{R}$ is a continuous function known as the zonal basis function (ZBF). In this case the interpolant is

$$s(\xi) = \sum_{j=1}^{N} \alpha_i \psi(g(\xi, \xi_j)), \quad \xi \in S^{d-1}. \tag{2.1}$$

Therefore, provided that the interpolation matrix

$$A_{ij} = \psi(g(\xi_i, \xi_j)), \quad 1 \leq i, j \leq N, \tag{2.2}$$

is non-singular, the interpolation conditions (1.1) define the real coefficients $\{\alpha_i : i = 1, \ldots, N\}$ uniquely.

Definition 2.1. *A continuous function* $\psi : [0, \pi] \to \mathbb{R}$ *is said to be strictly positive definite on* S^{d-1} ($\psi \in SPD(S^{d-1})$) *if, for any set* $\Xi = \{\xi_i\}_{i=1}^N$ *of distinct points on* S^{d-1}, *the quadratic form*

$$\alpha^T A \alpha = \sum_{j=1}^N \sum_{k=1}^N \alpha_j \alpha_k \psi(g(\xi_j, \xi_k)) \tag{2.3}$$

is positive on $\mathbb{R}^N \setminus \{0\}$.

If we choose $\psi \in SPD(S^{d-1})$, then the resulting interpolant (2.1) is unique since the corresponding interpolation matrix is, by definition, positive definite and hence non-singular.

Frequently one requires that an interpolant should reproduce the low order spherical harmonics. The ZBF interpolant s, given by (2.1), does not have this property and so it is often convenient to add to s a spherical harmonic of order k, which gives the form

$$s(\xi) = \sum_{j=1}^N \alpha_j \psi(g(\xi, \xi_j)) + \sum_{j=1}^M \beta_j \mathcal{Y}_j(\xi), \quad \xi \in S^{d-1}, \tag{2.4}$$

where $M = \dim \mathcal{H}_k(S^{d-1})$, and $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_M\}$ is a basis for $\mathcal{H}_k(S^{d-1})$.

The interpolation conditions (1.1) now provide N linear equations in $N + M$ unknowns, and so it is usual to impose M linear constraints

$$\sum_{j=1}^N \alpha_j \mathcal{Y}_i(\xi_j) = 0, \quad 1 \le i \le M, \tag{2.5}$$

to take up the extra degrees of freedom. Thus, we use the equations

$$\sum_{j=1}^N \alpha_j \psi(g(\xi_i, \xi_j)) + \sum_{j=1}^M \beta_j \mathcal{Y}_j(\xi_i) = f(\xi_i), \quad 1 \le i \le N, \tag{2.6}$$

and

$$\sum_{j=1}^N \alpha_j \mathcal{Y}_i(\xi_j) \qquad\qquad = 0, \qquad 1 \le i \le M, \tag{2.7}$$

or equivalently, we have the linear system

$$\begin{pmatrix} A & Y \\ Y^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{2.8}$$

where A is as in (2.2) and $Y \in \mathrm{I\!R}^{N \times M}$ is given by

$$Y_{ij} = \mathcal{Y}_j(\xi_i), \quad \text{where } 1 \leq i \leq N, \text{ and } 1 \leq j \leq M. \tag{2.9}$$

Thus a unique augmented ZBF interpolant exists if and only if the augmented interpolation matrix

$$\hat{A} = \begin{pmatrix} A & Y \\ Y^T & 0 \end{pmatrix} \tag{2.10}$$

is non-singular.

Definition 2.2. *For any set $\Xi = \{\xi_i\}_{i=1}^N$ of distinct data points on S^{d-1} we consider the following subspace*

$$W_{m-1} = \left\{ \alpha \in \mathrm{I\!R}^N : \sum_{i=1}^N \alpha_i \mathcal{Y}(\xi_i) = 0 \text{ for all } \mathcal{Y} \in \mathcal{H}_{m-1}(S^{d-1}) \right\}. \tag{2.11}$$

A continuous function $\psi : [0, \pi] \to \mathrm{I\!R}$ is said to be conditionally strictly positive definite of order $m \in \mathrm{I\!N}$ on S^{d-1}, ($\psi \in CSPD_m(S^{d-1})$) if the quadratic form (2.3) is positive on $W_{m-1} \setminus \{0\}$.

Any function $\psi \in CSPD_m(S^{d-1})$ can be used to provide an augmented ZBF interpolant of the form (2.4) with $k = m-1$. However, in order to guarantee the uniqueness of such a solution we impose some restrictions on the data points.

Unisolvency: *Let m be a positive integer and let $M = \dim \mathcal{H}_{m-1}(S^{d-1})$. A set of distinct points $\Xi = \{\xi_i\}_{i=1}^M$ is said to be $\mathcal{H}_{m-1}(S^{d-1})-$unisolvent if the only element of $\mathcal{H}_{m-1}(S^{d-1})$ to vanish at each ξ_i is the zero spherical harmonic.*

The following theorem, see [13], establishes the non-singularity of the augmented interpolation matrix $\hat{A}$ in the case of $\psi \in CSPD_m(S^{d-1})$.

Theorem 2.3. *Let $\psi \in CSPD_m(S^{d-1})$ and let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of N distinct data points in S^{d-1} such that*

(i) $N \geq M = \dim \mathcal{H}_{m-1}(S^{d-1})$,

(ii) $\{\xi_i\}_{i=1}^M$ is an $\mathcal{H}_{m-1}(S^{d-1})-$unisolvent subset.

Then the augmented interpolation matrix $\hat{A}$, given by (2.10), is non-singular.

Using the work of Schoenberg [19], and extensions thereof [6], we can formulate the following theorem.

Theorem 2.4. *If $\psi \in CSPD_m(S^{d-1})$, then it has the following form*

$$\psi(\theta) = \sum_{k=0}^{\infty} a_k P_{k,d}(\cos \theta), \tag{2.12}$$

where

$$a_k \geq 0 \quad for \quad k \geq m \quad and \quad \sum_{k=0}^{\infty} a_k < \infty, \qquad (2.13)$$

where $\{P_{k,d}\}$ denote the $d-$dimensional Legendre polynomials, given by (1.9).

Remark 2.5. *(i) In view of Theorem 2.4 we choose to consider each ZBF ψ as a function of the inner product, $\xi^T \eta$, since $\cos(g(\xi, \eta)) = \xi^T \eta$.*

(ii) Throughout this paper we shall take $\psi \in CSPD_0(S^{d-1})$ to mean $\psi \in SPD(S^{d-1})$. Further, if $\psi \in CSPD_m(S^{d-1})$ with $m > 0$, then we shall assume without loss that $a_k = 0$ for $0 \leq k \leq m - 1$.

The complete characterisation of the class of functions of the form (2.12) satisfying (2.13) that are $CSPD_m(S^{d-1})$ has been investigated by several researchers see [14], [17] and [18]. The most recent result is due to Chen, Menegatto and Sun [4] who show that, for $d \geq 3$, a necessary and sufficient condition is that the set $\{k \in \mathbb{N}_0 \setminus \{0, 1, \ldots, m-1\} : a_k > 0\}$ must contain infinitely many odd and infinitely many even integers. The case of $d = 2$ remains an open problem and so we will only consider basis functions $\psi \in CSPD_m(S^{d-1})$ for which $a_k > 0$ for $k \geq m$.

3   A variational theory

For every $\psi \in SCPD_m(S^{d-1})$ we can associate a zonal kernel $\Psi(\xi, \eta) = \psi(\xi^T \eta)$. This, in turn, has a unique spherical Fourier expansion, given by

$$\Psi(\xi, \eta) = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta), \qquad (3.14)$$

where the $\hat{c}_k$ denote the spherical Fourier coefficients of Ψ. These are related to the Legendre coefficients of ψ by (1.13). Furthermore, each sequence $\{\hat{c}_k\}_{k \geq m}$ possesses a certain decay rate as $k \to \infty$. In particular, we say that

1. ψ has $\alpha-$Fourier decay if there exists positive constants A_1, A_2 such that

$$A_1(1+k)^{-(d-1+\alpha)} \leq \hat{c}_k \leq A_2(1+k)^{-(d-1+\alpha)}, \quad \alpha > 0, \quad k \geq m. \quad (3.15)$$

2. ψ has $e-$Fourier decay if the $\hat{c}_k$ decay at an exponentially fast rate.

Defintion 3.6. *Let $\psi \in SCPD_m(S^{d-1})$ and let $\{\hat{c}_k\}_{k \geq m}$ denote the spherical Fourier coefficients of its associated zonal kernel (3.14). We define the native space of ψ to be*

$$H_{\psi,m} := \left\{ f \in L_2(S^{d-1}) : |f|^2_{\psi,m} = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k} < \infty \right\}, \qquad (3.16)$$

where $|\cdot|_{\psi,m}$ is a (semi-)norm induced via the (semi-)inner product

$$(f,g)_{\psi,m} = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{\hat{f}_{k,l}\hat{g}_{k,l}}{\hat{c}_k}. \qquad (3.17)$$

Note 1. If $m = 0$ then $|\cdot|_{\psi,0}$ is a norm which we rewrite as $\|\cdot\|$. Furthermore, if ψ has $\alpha-$Fourier decay then $H_{\psi,0}$ is norm equivalent to the Sobolev space $W_2^{\beta}(S^{d-1})$ where $\beta = \frac{d-1+\alpha}{2}$, that is, there exists constants $0 < k_{eq} < K_{eq}$, such that

$$k_{eq}\|\cdot\|_{W_2^{\beta}(S^{d-1})} \leq \|\cdot\|_{\psi} \leq K_{eq}\|\cdot\|_{W_2^{\beta}(S^{d-1})}. \qquad (3.18)$$

Note 2. For $m > 0$, we can use the fact that $\mathcal{H}_{m-1}(S^{d-1})$ is a finite dimensional Hilbert space to modify $(\cdot,\cdot)_{\psi,m}$ so that it becomes a genuine inner product. In particular, if we assume that $\{\xi_1, \ldots, \xi_M\}$ is a $\mathcal{H}_{m-1}(S^{d-1})-$unisolvent set, then a suitable inner product is

$$< f,g >_{\mathcal{H}_{m-1}(S^{d-1})} = \sum_{i=1}^{M} f(\xi_i)g(\xi_i), \quad f,g \in \mathcal{H}_{m-1}(S^{d-1}). \qquad (3.19)$$

Moreover, we have that

$$< f,g >_{\psi} = < f,g >_{\mathcal{H}_{m-1}(S^{d-1})} + (f,g)_{\psi,m} \qquad (3.20)$$

is an inner-product for $H_{\psi,m}$. Indeed, we have the following new definition.

Definition 3.7. *Let $m > 0$ and let $\psi \in CSPD_m(S^{d-1})$. We define the **normed** native space of ψ by*

$$H_{\psi} = \left\{ f \in L_2(S^{d-1}) : \|f\|_{\psi} < \infty \right\}, \qquad (3.21)$$

where $\|\cdot\|_{\psi}$ is the norm induced by the inner product (3.20).

Note 3. Since all norms are equivalent on finite dimensional spaces, we can use the same arguments as in Note 1 to deduce that, if ψ has $\alpha-$Fourier decay

then H_ψ is norm equivalent to $W_2^\beta(S^{d-1})$, where $\beta = \frac{d-1+\alpha}{2}$. In particular, we can use the Sobolev embedding theorem to conclude that H_ψ is a Hilbert space of continuous functions.

The importance of the native space of a basis fuction $\psi \in SCPD_m(S^{d-1})$ is well illustrated by Levesley et al in [13], where it is shown that, given any $f \in H_\psi$, the solution to the following variational problem

$$\text{minimise}\left\{\|s\|_\psi : s \in H_\psi \text{ and } s(\xi_i) = f(\xi_i) \ 1 \leq i \leq N\right\}, \tag{3.22}$$

is precisely the unique $\psi-$based ZBF interpolant. This variational problem is precisely the same as finding the optimal interpolant in a Hilbert space, such problems are well understood and were studied in the late 1950s by Golomb and Weinberger [9]. The real power of the variational approach lies in the fact that the original Hilbert space techniques from [9] can be applied to provide useful pointwise error bounds. Specifically, for a given $f \in H_\psi$, the error of its $\psi-$based ZBF interpolant s_f can be bounded by an estimate of the form

$$|s_f(\xi) - f(\xi)| \leq P_\psi(\xi) \cdot \|s_f - f\|_\psi, \quad \xi \in S^{d-1}. \tag{3.23}$$

The factor $P_\psi(\xi)$ is called the *power function* of $\psi \in CSPD_m(S^{d-1})$ and has the following explicit form

$$P_\psi(\xi) = \left(\sum_{i=1}^N \sum_{j=1}^N \gamma_i \gamma_j \psi(\xi_i^T \xi_j) - 2 \sum_{i=1}^N \gamma_i \psi(\xi^T \xi_i) + \psi(1) \right)^{1/2},$$

where the coefficients $\{\gamma_i \in \mathbb{R} : i = 1, \ldots, N\}$ are chosen to satisfy

$$\mathcal{Y}(\xi) = \sum_{i=1}^N \gamma_i \mathcal{Y}(\xi_i), \quad \text{for all } \mathcal{Y} \in \mathcal{H}_{J-1}(S^{d-1}), \tag{3.24}$$

where $J \geq m$ is a fixed integer. Stated in this way, it is clear that a close investigation of P_ψ, and especially the choice of the γ_i, ought to provide an insight into the accuracy of the ZBF interpolation method. Indeed this strategy is employed, in quite different ways, by Jetter et al [12] and also by von Golitschek and Light [7] to provide error bounds of the form

$$|s_f(\xi) - f(\xi)| \leq \beta(h) \cdot \|s_f - f\|_\psi, \quad \xi \in S^{d-1}, \tag{3.25}$$

where $\beta(h)$ is a function which tends to zero as $h \to 0$.

We remark that the error bound (3.23) may be viewed as a specific instance of the following more general result.

Proposition 3.8. *Let $\psi \in CSPD_m(S^{d-1})$ and let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points on S^{d-1}. Consider the subspace*

$$Z_\psi = \{f \in H_\psi : f(\xi_i) = 0 \quad i = 1, \ldots, N\},$$

then

$$|f(\xi)| \leq P_\psi(\xi) \cdot \|f\|_\psi, \quad \text{for all } f \in Z_\psi \text{ and } \xi \in S^{d-1}. \tag{3.26}$$

So far in this paper we have used the mesh-norm h to measure the relative density of a set of data points $\Xi = \{\xi_i\}_{i=1}^N$ in S^{d-1}. Geometrically speaking, h represents the radius of the largest spherical cap (open geodesic ball) which can be placed on S^{d-1} without covering any $\xi_i \in \Xi$. In [7], von Golitschek and Light use the height h_d of the maximal spherical cap as an alternative mesh-norm. That is, they define h_d to be the smallest number such that

$$\inf_{\eta \in S^{d-1}} \max\{\eta^T \xi_i : \xi_i \in \Xi\} > 1 - h_d, \tag{3.27}$$

is satisfied. We shall call h_d the "dot product" mesh norm of Ξ. Using some elementary trigonometry we can show that $h_d = 2\sin^2(h/2)$. Furthermore, if $h \in (0, 2\pi/3)$ then we can apply the small angle result for $\sin(h/2)$ to give

$$\frac{h^2}{8} \leq h_d \leq \frac{h^2}{2} \tag{3.28}$$

that is, h_d is equivalent to h^2. The idea of using the dot product as an alternative measure of distance will prove to be a useful one.

Definition 3.9. *For every $\xi \in S^{d-1}$ we define an associated a dot-product distance function*

$$d_\xi : S^{d-1} \to [-1, 1], \quad \text{given by } d_\xi(\eta) = \xi^T \eta.$$

Furthermore, we can define a dot product neighbourhood of ξ by

$$N(\xi, r_d) = \{\eta \in S^{d-1} : d_\xi(\eta) > 1 - r_d\}, \quad \text{where } r_d \in (0, 1). \tag{3.29}$$

The following crucial result is quoted from [7].

Lemma 3.10. *Let J be a fixed positive integer and let $\Xi = \{\xi_1, \ldots, \xi_N\}$ denote a set of N distinct data points in S^{d-1} with dot product mesh-norm h_d. There is a number $h_0 \in (0, 1)$ such that if $h_d < h_0$, and $\xi \in S^{d-1}$, then there exist coefficients $\{\gamma_i\}_{i=1}^N$ such that*

1. $\mathcal{Y}(\xi) = \sum_{i=1}^{N} \gamma_i \mathcal{Y}(\xi_i)$, *for all* $\mathcal{Y} \in \mathcal{H}_{J-1}(S^{d-1})$,

2. *there exists a constant K_1 (independent of ξ and h_d) such that if $\xi_i \notin N(\xi, K_1 h_d)$, then $\gamma_i = 0$, and*

3. *there exists a constant K_2 (independent of ξ and h_d) such that $\sum_{i=1}^{N} |\gamma_i| \leq K_2$.*

With this preparation the following result can be established.

Theorem 3.11. *Let $\psi \in CSPD_m(S^{d-1})$ have $\alpha-$Fourier decay and let $\Xi = \{\xi_i\}_{i=1}^{N}$ denote a set distinct points on S^{d-1} with mesh-norm h. Set*

$$J = \max\left\{m, \frac{\lceil \alpha + 1 \rceil}{2}\right\}, \tag{3.30}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$, and assume that the dot product mesh-norm h_d (3.27) of Ξ satisfies

$$\frac{1}{(K+1)^2} \leq h_d < \frac{1}{K^2}, \tag{3.31}$$

where $K > J$ is a positive integer. Let $f \in H_\psi$ and s_f denote its unique ZBF interpolant. Then, for any $\xi \in S^{d-1}$, we have

$$|f(\xi) - s_f(\xi)| \leq \mathcal{C} \cdot h^{\alpha/2} \cdot \|f - s_f\|_\psi, \tag{3.32}$$

where $\mathcal{C}$ is a positive constant independent of h.

Proof. For a full proof of this result see [15], Theorem 2. For a brief sketch, we note that choice of integer J allows us to evoke Lemma 3.10 to provide, for any $\xi \in S^{d-1}$, a neighbourhood $N(\xi, K_1 h_d)$ and a set of local coefficients $\{\gamma_i\}_{i \in I_{loc}}$, where $I_{loc} := \{i : \xi_i \in \Xi \cap N(\xi, K_1 h_d)\}$, which satisfy condition (3.24). Furthermore, these coefficients can be used to define a local power function, $P_{\psi,loc}$ say, which by (3.23), provides the pointwise error bound

$$|s_f(\xi) - f(\xi)| \leq P_{\psi,loc}(\xi) \cdot \|s_f - f\|_\psi, \quad \xi \in S^{d-1}. \tag{3.33}$$

It then remains to show that $P_{\psi,loc}$ can be bounded above by a constant multiplied by $h^{\alpha/2}$. $\qquad\square$

We close this section by providing two important properties of the ZBF interpolant, both of which can be inferred from the theory of optimal interpolation in a Hilbert space [9].

Lemma 3.12. *Let $\psi \in CSPD_m(S^{d-1})$. For a given $f \in H_\psi$ let s_f denote its unique $\psi-$based ZBF interpolant, then we have*

$$(i) \ \|f - s_f\|_\psi^2 \ = < f, f - s_f >_\psi, \quad (ii) \ \|f - s_f\|_\psi \leq \|f\|_\psi.$$

4 Global error estimates

In this section we generalise techniques dating back to Duchon [5], from his study of the accuracy of interpolation using D^m-splines in Euclidean space. The crucial ingredients for a Duchon framework for the sphere are as follows

(i) A suitable quasi-uniform mesh of data points for the sphere.

(ii) A suitable Sobolev extension operator for the sphere.

(iii) A spherical version of Duchon's inequality.

The technical effort required to establish these items is quite considerable. In view of this, we shall simply state the key results and refer the reader to our accompanying paper [11] for full details.

4.1 The key results

$\diamond$ **A quasi-uniform mesh for the sphere:**

Lemma 4.1. *Let $d \geq 2$ be an integer and set $M = 2\sqrt{d-1}$. Let $\theta \in (0, \pi - \delta_d)$, where $\delta_d = \frac{1}{4d^{3/2}} < \frac{\pi}{32}$. Let M_1 be an arbitrary positive number and set*

$$h_0 := \min\left\{ \frac{\theta}{M + M_1}, \ 1\right\}. \tag{4.1}$$

Then, for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset S^{d-1}$ such that

$$S^{d-1} = \cup_{z \in Z_h} G(z, Mh).$$

Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. There exists a positive integer Q independent of h such that

$$\sum_{z \in Z_h} F_{G(z, \overline{M}h)} \leq Q, \quad \text{where} \ \ \overline{M} = 2\sqrt{d-1} + M_1. \tag{4.2}$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-(d-1)}$, where C_Q is independent of h.

◇ **A Sobolev extension theorem for the sphere:**

Theorem 4.2. *Let $z \in S^{d-1}$ and $\Xi = \{\xi_i\}_{i=1}^{N}$ denote a set of distinct points on S^{d-1}. Let $\beta \in [k, k+1]$, where $k > \frac{d-1}{2}$ is a positive integer. There exists positive numbers $\mathcal{R}_0$ and $C_{\mathcal{A}}$ such that if we let $M_1 > \max\{\mathcal{R}_0 - 2\sqrt{d-1}, 0\}$ be a fixed positive number and let*

$$h_0 = C_{\mathcal{A}}/(3\overline{M}) \quad \text{where} \ \ \overline{M} = 2\sqrt{d-1} + M_1, \tag{4.3}$$

then, assuming that Ξ has mesh norm $h \in (0, h_0)$, there exists an extension operator $E_{G(z, \overline{M}h)} : W_2^{\beta}(G(z, \overline{M}h)) \to W_2^{\beta}(S^{d-1})$ satisfying

1) $(E_{G(z, \overline{M}h)} f)|_{G(z, \overline{M}h)} = f$, for all $f \in W_2^{\beta}(G(z, \overline{M}h)$,

2) there exists a positive constant $\mathcal{K}$, independent of h and z such that

$$\|E_{G(z, \overline{M}h)} f\|_{W_2^{\beta}(S^{d-1})} \leq \mathcal{K} \cdot \|f\|_{W_2^{\beta}(G(z, \overline{M}h))},$$

for all $f \in W_2^{\beta}(G(z, \overline{M}h))$ such that $f(\xi) = 0$ for $\xi \in \Xi \cap G(z, \overline{M}h)$.

◇ **A spherical version of Duchon's inequality:**

Theorem 4.3. *Let $\beta > 0$ and let M_1 be any positive number. Set h_0 to be as in (4.3), let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for S^{d-1} from Lemma 4.1. Then, for any $f \in W_2^{\beta}(S^{d-1})$, we have*

$$\sum_{z \in Z_h} \|f\|_{W_2^{\beta}(G(z, \overline{M}h))}^2 \leq Q \|f\|_{W_2^{\beta}(S^{d-1})}^2 \tag{4.4}$$

where Q is the constant (independent of h) from Lemma 4.1.

We now derive the first L_p error bounds for ZBF interpolation.

Theorem 4.4. *Assume that $\psi \in CSPD_m(S^{d-1})$ has $\alpha-$Fourier decay. Let Ξ denote a set of distinct data points on S^{d-1} with mesh-norm $h \in (0, \pi/6)$ (1.2) and dot product mesh norm h_d (3.27). In addition, suppose that*

1. J and K are the integers as defined in Theorem 3.11, and K_1 denotes the neighbourhood constant (corresponding to J) from Lemma 3.10.

2. The dot product mesh-norm satisfies $h_d < \frac{3}{2K_1}$.

3. $\mathcal{R}_0$ and $C_{\mathcal{A}}$ are the constants from Theorem 4.2 corresponding to $\beta = \frac{\alpha+d-1}{2}$.

4. For a given constant

$$M_1 > \max\{\mathcal{R}_0 - 2\sqrt{d-1}, 2\sqrt{K_1}\}, \qquad (4.5)$$

the mesh-norm h satisfies

$$0 < h < h_0 = C_{\mathcal{A}}/(3\overline{M}) \quad \text{where } \overline{M} = 2\sqrt{d-1} + M_1.$$

As usual, let s_f denote the unique ZBF interpolant to a target function $f \in H_\psi$. There exists a constant C, independent of h, such that

$$\|f - s_f\|_{L_p(S^{d-1})} \leq C \cdot h^{\frac{\alpha}{2}+\frac{d-1}{p}} \|f - s_f\|_\psi, \quad \text{for } p \in [2, \infty) \qquad (4.6)$$

and

$$\|f - s_f\|_{L_p(S^{d-1})} \leq C \cdot h^{\frac{\alpha}{2}+\frac{d-1}{2}} \|f - s_f\|_\psi, \quad \text{for } p \in [1, 2). \qquad (4.7)$$

Proof. Using Lemma 4.1 we can deduce that

$$\|f - s_f\|^p_{L_p(S^{d-1})} = \int_{S^{d-1}} |(f - s_f)(\xi)|^p d\omega_{d-1}(\xi)$$

$$\leq \sum_{z \in Z_h} \int_{G(z,Mh)} |(f - s_f)(\xi)|^p d\omega_{d-1}(\xi), \quad \text{for } M = 2\sqrt{d-1}.$$

The function $f - s_f$ is continuous on $\overline{G(z, Mh)}$ and, as this is a compact subset of S^{d-1}, there exists a point $\xi_z \in \overline{G(z, Mh)}$ at which $f - s_f$ attains its maximum. This observation allows us to write

$$\|f - s_f\|^p_{L_p(S^{d-1})} \leq \sum_{z \in Z_h} |(f - s_f)(\xi_z)|^p \int_{G(z,Mh)} d\omega_{d-1}(\xi)$$

$$\leq C_{P_1} \cdot h^{d-1} \sum_{z \in Z_h} |(f - s_f)(\xi_z)|^p, \qquad (4.8)$$

where C_{P_1} is independent of h.

Now, rather than consider $f - s_f$, we choose instead to consider the restriction $f - s_f|_{G(z,\overline{M}h)}$ where $\overline{M} = 2\sqrt{d-1} + M_1$, for some $M_1 \geq 0$. Since the native space H_ψ is norm-equivalent to the Sobolev space $W_2^\beta(S^{d-1})$, we have that $f - s_f|_{G(z,\overline{M}h)}$ belongs to the local Sobolev space $W_2^\beta(G(z,\overline{M}h))$, for each $z \in Z_h$. In choosing a suitable value for M_1, and hence $\overline{M}$, we must take into account the following conditions.

(a) In order to employ Theorem 3.11 to provide pointwise error estimates, we require that each $G(z,\overline{M}h)$ must contain the dot product neighbourhood $N(\xi_z, K_1 h_d)$.

(b) In order to apply the Sobolev extension operator to $f - s_f|_{G(z,\overline{M}h)} \in W_2^\beta(G(z,\overline{M}h))$, we require that $\overline{M}h \in (\mathcal{R}_0 h, C_\mathcal{A}/3)$.

We note that the choice of M_1 given by (4.5) is sufficient to ensure condition **(b)** is satisfied. It remains to show that, with this choice, condition **(a)** is also satisfied.
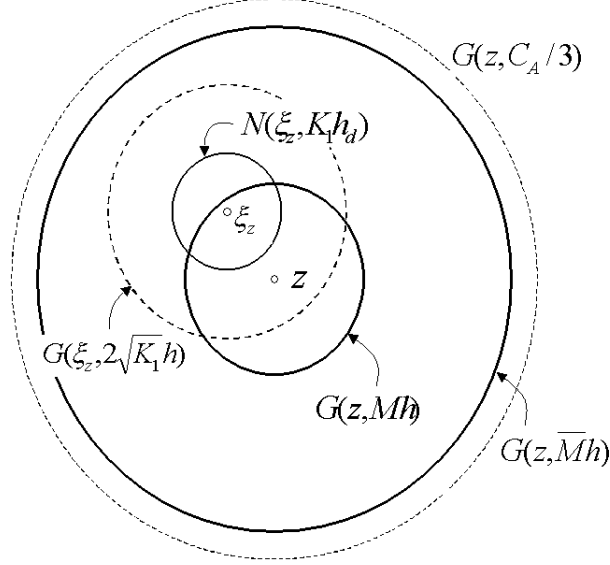
Figure 1: Illustration of the nesting of the key neighbourhoods

For any ξ_z, the neighbourhood $N(\xi_z, K_1 h_d)$ can also be viewed, in more familiar terms, as an open geodesic ball $G(\xi_z, \theta)$, where θ satisfies $\sin^2(\theta/2) = K_1 h_d/2$. Now since $h_d < 3/(2K_1)$, we have that $\theta \in (0, 2\pi/3)$, thus we can apply the small angle result for $\sin(\theta/2)$, followed by the mesh-norm equivalence relation (3.28) to deduce that

$$\frac{\theta}{2\sqrt{2}} \le \sqrt{K_1 h_d} \le \sqrt{K_1} \cdot \frac{h}{\sqrt{2}}.$$

In particular, this shows that

$$
\begin{aligned}
N(\xi_z, K_1 h_d) = G(\xi_z, \theta) &\subset G(\xi_z, 2\sqrt{K_1} h) \\
&\subset G(z, (M + 2\sqrt{K_1})h) \\
&\subset G(z, (M + M_1)h) = G(z, \overline{M}h) \\
&\subset G(z, C_{\mathcal{A}}/3), \qquad \text{(see Figure 1)}.
\end{aligned}
$$

Thus the choice of M_1, given by (4.5), is sufficient to ensure that condition (**a**) is also satisfied.

Let $v_z = f - s_f|_{G(z,\overline{M}h)}$ then, using the Sobolev extension operator, we have

E1. $E_{G(z,\overline{M}h)} v_z \in W_2^\beta(S^{d-1})$.

E2. $E_{G(z,\overline{M}h)} v_z(\xi) = 0$ for all $\xi \in \Xi \cap G(z,\overline{M}h)$.

E3. Using part 2 of Theorem 4.2, there exists a constant $\mathcal{K}$, independent of h and z such that $\|E_{G(z,\overline{M}h)} v_z\|_{W_2^\beta(S^{d-1})} \leq \mathcal{K} \cdot \|v_z\|_{W_2^\beta(G(z,\overline{M}h))}$.

Since condition (**a**) is satisfied, we have that

$$\Xi_{\xi_z} = \Xi \cap N(\xi_z, K_1 h_d) \subset \Xi \cap G(z,\overline{M}h) = \Xi_z.$$

Thus, the optimal power function of ψ, based upon Ξ_z and evaluated at the point ξ_z, can be bounded above by the local power function $P_{\psi,loc}(\xi_z)$. Furthermore, applying Proposition 3.8, (3.33), (3.18) and **E3** respectively, yields

$$|(f - s_f)(\xi_z)| = |E_{G(z,\overline{M}h)} v_z(\xi_z)| \leq P_{\psi,loc}(\xi_z)\|E_{G(z,\overline{M}h)} v_z\|_\psi$$

$$\leq P_{\psi,loc}(\xi_z) K_{eq} \|E_{G(z,\overline{M}h)} v_z\|_{W_2^\beta(S^{d-1})}$$

$$\leq \mathcal{C} K_{eq} \cdot h^{\frac{\alpha}{2}} \|E_{G(z,\overline{M}h)} v_z\|_{W_2^\beta(S^{d-1})}$$

$$\leq \mathcal{K} \mathcal{C} K_{eq} \cdot h^{\frac{\alpha}{2}} \|v_z\|_{W_2^\beta(G(z,\overline{M}h))}.$$

Substituting this into (4.8) gives

$$\|f - s_f\|_{L_p(S^{d-1})}^p \leq C_{P_2} \cdot h^{(\frac{\alpha p}{2}+d-1)} \sum_{z \in Z_h} \|f - s_f|_{G(z,\overline{M}h)}\|_{W_2^\beta(G(z,\overline{M}h))}^p,$$

where $C_{P_2} = C_{P_1}(\mathcal{K}\mathcal{C}K_{eq})^p$ is independent of h.

For $p \geq 2$ we use Jensen's inequality $\sum_{i=1}^N a_i^p \leq (\sum_{i=1}^N a_i^2)^{p/2}$ [2], followed by Theorem 4.3, and (3.18) to give

$$\|f - s_f\|_{L_p(S^{d-1})}^p \leq C_{P_2} \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot \left(\sum_{z \in Z_h} \|(f - s_f)|_{G(z,\overline{M}h)}\|_{W_2^\beta(G(z,\overline{M}h))}^2 \right)^{p/2}$$

$$\leq C_{P_2} Q^{\frac{p}{2}} \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot \left(\|f - s_f\|_{W_2^\beta(S^{d-1})}^2 \right)^{p/2}$$

$$\leq (C_{P_2} Q^{\frac{p}{2}} k_{eq}^{-p}) \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot \|f - s_f\|_{H_\psi}^p.$$

Finally, taking the pth root gives

$$\|f - s_f\|_{L_p(S^{d-1})} \leq \ C \cdot h^{\frac{\alpha}{2} + \frac{d-1}{p}} \|f - s_f\|_\psi,$$

where $C = (C_{P_2} Q^{\frac{p}{2}} k_{eq}^{-p})^{1/p}$ is independent of h.

For $p \in [1,2)$ we execute the same arguments as above, however we replace Jensen's inequality with $\sum_{i=1}^N a_i^p \leq N^{1-\frac{p}{2}} \left(\sum_{i=1}^N a_i^2 \right)^{p/2}$ [2]. Further, we use the fact that the cardinality of Z_h is bounded by $C_Q h^{-(d-1)}$, see Lemma 4.1, to deduce that

$$\|f - s_f\|_{L_p(S^{d-1})}^p \leq C_{P_2} \cdot h^{(\frac{\alpha p}{2} + d - 1)} \cdot \left(\sum_{z \in Z_h} \|(f - s_f)|_{G(z, \overline{M}h)}\|_{W_2^\beta(G(z, \overline{M}h))}^2 \right)^{p/2}$$

$$\leq C_{P_2} Q^{\frac{p}{2}} C_Q \cdot h^{p(\frac{\alpha + d - 1}{2})} \cdot \left(\|f - s_f\|_{W_2^\beta(S^{d-1})}^2 \right)^{p/2}$$

$$= (C_{P_2} Q^{\frac{p}{2}} k_{eq}^{-p}) C_Q \cdot h^{p(\frac{\alpha + d - 1}{2})} \cdot \|f - s_f\|_\psi^p.$$

Finally, taking the pth root provides

$$\|f - s_f\|_{L_p(S^{d-1})} \leq \ C \cdot h^{\frac{\alpha + d - 1}{2}} \|f - s_f\|_\psi,$$

where $C = (C_{P_2} Q^{\frac{p}{2}} k_{eq}^{-p} C_Q)^{1/p}$ is independent of h. $\qquad\qquad\square$

At first glance it is tempting to "tidy up" the error result (4.6) by employing the optimality bound $\|f - s_f\|_\psi \leq \|f\|_\psi$, from Lemma 3.12 (ii). This is a perfectly valid procedure, however we will show that an improved bound is available, provided that f belongs to a certain subspace of H_ψ, which we shall denote as $H_{\psi * \psi}$. Once this improved bound is established we will use it to improve the L_p−convergence order in (4.6) for target functions $f \in H_{\psi * \psi}$.

Definition 4.5. *Let $\psi \in CSPD_m(S^{d-1})$ have $\alpha−$Fourier decay and let Ψ denote its corresponding zonal kernel. We define the convolution kernel of Ψ by*

$$(\Psi * \Psi)(\xi, \eta) := \int_{S^{d-1}} \Psi(\xi, \nu) \Psi(\nu, \eta) d\omega_{d-1}(\nu), \quad \xi, \eta \in S^{d-1}.$$

It is more revealing to work in terms of Fourier expansions since we have

$$\Psi(\xi, \eta) = \sum_{k=m}^\infty \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta) \Rightarrow (\Psi * \Psi)(\xi, \eta) = \sum_{k=m}^\infty \sum_{l=1}^{N_{k,d}} \hat{c}_k^2 \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta).$$

This observation allows us to define a convolution native space by

$$H_{\psi*\psi,m} = \left\{ f \in L_2(S^{d-1}) : |f|_{\psi*\psi,m} = \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k^2} \right)^{1/2} < \infty \right\}.$$

The observations made in Section 2, regarding native spaces, also apply to convolution native spaces. In particular, we can define a normed native space $(H_{\psi*\psi}, \|\cdot\|_{\psi*\psi})$ and conclude that

$$(H_{\psi*\psi}, \|\cdot\|_{\psi*\psi}) \cong W_2^{2\beta}(S^{d-1}) \subset W_2^{\beta}(S^{d-1}) \cong (H_\psi, \|\cdot\|_\psi), \qquad (4.9)$$

where $\beta = \frac{\alpha+d-1}{2}$ and where $\cong$ denotes norm equivalence.

Lemma 4.6. *For a given $f \in H_{\psi*\psi}$, let s_f denote its unique $\psi-$based ZBF interpolant. Then*

$$\|f - s_f\|_\psi^2 \leq \|f\|_{\psi*\psi} \cdot \|f - s_f\|_{L_2(S^{d-1})}. \qquad (4.10)$$

Proof. Using Lemma 3.12 (*i*), the definition of $< \cdot, \cdot >_\psi$ and an application of the Cauchy-Schwarz inequality respectively, gives

$$\|f - s_f\|_\psi^2 = <f, f - s_f>_\psi = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{\hat{f}_{k,l} \cdot (\hat{f}_{k,l} - (\hat{s_f})_{k,l})}{\hat{c}_k}$$

$$\leq \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k^2} \right)^{1/2} \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} (\hat{f}_{k,l} - (\hat{s_f})_{k,l})^2 \right)^{1/2}$$

$$\leq \|f\|_{\psi*\psi} \cdot \|f - s_f\|_{L_2(S^{d-1})}.$$

$\square$

With this in place we can provide the following improved error bound.

Theorem 4.7. *Assume the same set up as in Theorem 4.4 and assume further that the target function f belongs to $H_{\psi*\psi}$. Then we have*

$$\|f - s_f\|_{L_p(S^{d-1})} \leq C^2 \cdot h^{\alpha+\frac{d-1}{2}+\frac{d-1}{p}} \|f\|_{\psi*\psi}, \quad \text{for } p \in [2, \infty), \qquad (4.11)$$

and

$$\|f - s_f\|_{L_p(S^{d-1})} \leq C^2 \cdot h^{\alpha+d-1} \|f\|_{\psi*\psi}, \quad \text{for } p \in [1, 2], \qquad (4.12)$$

where C is the constant, independent of h, from Theorem 4.4.

Proof. Since $f \in H_{\psi*\psi} \subset H_\psi$ we have, from Theorem 4.4 with $p = 2$, that

$$\|f - s_f\|_{L_2(S^{d-1})} \leq C \cdot h^{\frac{\alpha}{2} + \frac{d-1}{2}} \|f - s_f\|_\psi,$$

substituting this into (4.10) gives

$$\|f - s_f\|_\psi^2 \leq C h^{\frac{\alpha}{2} + \frac{d-1}{2}} \|f\|_{\psi*\psi} \|f - s_f\|_\psi,$$

cancelling the factor $\|f - s_f\|_\psi$ gives

$$\|f - s_f\|_\psi \leq C \cdot h^{\frac{\alpha}{2} + \frac{d-1}{2}} \|f\|_{\psi*\psi}. \tag{4.13}$$

Substituting this inequality into the results of Theorem 4.4, namely (4.6) and (4.7), proves the theorem. $\qquad\square$

Corollary 4.8. *Assuming the same set up as in Theorem 4.7, we have*

$$\|f - s_f\|_{L_\infty(S^{d-1})} \leq C \cdot h^{\alpha + \frac{d-1}{2}} \|f\|_{\psi*\psi}. \tag{4.14}$$

where C is a positive constant independent of h.

Proof. Since $f \in H_{\psi*\psi} \subset H_\psi$, we can appeal to Theorem 3.11 to deduce that there exists a constant $\mathcal{C}$ independent of h such that

$$\|f - s_f\|_{L_\infty(S^{d-1})} \leq \mathcal{C} \cdot h^{\alpha/2} \cdot \|f - s_f\|_\psi.$$

The proof is completed by substituting (4.13) into the above. $\qquad\square$

5 Conclusions

In [10], a numerical investigation into the performance of the ZBF method is presented. In particular, the numerical evidence strongly suggests that if $\psi \in CSPD_m(S^{d-1})$ has $\alpha-$Fourier decay and $f \in H_{\psi*\psi}$, then the optimal L_p-error bound has the form

$$\|f - s_f\|_{L_p(S^{d-1})} \leq C \cdot h^{\alpha + d - 1} \|f\|_{\psi*\psi}, \quad p \in [1, \infty], \tag{5.1}$$

for some constant C independent of h. Comparing this result with our theoretical error bounds, (4.11) and (4.12), we find that we have complete agreement in the case of $p \in [1, 2]$. However, for $p > 2$, there is gap between the theoretical bound and the numerically observed bound. Indeed, the authors believe that the task of bridging this gap that is, replacing the factor $\frac{d-1}{p}$ in (4.11) with $\frac{d-1}{2}$, is a challenging puzzle and one which deserves further investigation.

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